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Koiter's Modified Energy Functional for Circular Cylindrical Shells

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Nomenclature

E	= modulus of elasticity of shell material
h	= shell thickness
R	= shell radius
u,v,w	= shell displacements in longitudinal, circumferential and radial directions, respectively
$(\ )',(\ )^{\cdot}$	= derivatives of displacements with respect to longitudinal or circumferential coordinates, respectively
ν	= Poisson's ratio
dS	= infinitesimal element of shell middle surface

Introduction

THE search for simple and convenient yet accurate equations to describe the behavior of circular cylindrical shells led Donnell[1] to make approximations which Hoff[2] has shown limit the accuracy of Donnell's equations to relatively short shells. Seeking equations of more general value, Morley[3] proposed but did not rigorously justify a modification of the Donnell equations which appeared to preserve the convenience of the originals while possessing an accuracy comparable to Flügge's[4] over a broad range of practical shell proportions. Now Koiter has suggested a rational modification to his energy functional[5] which yields, by way of variational methods, equilibrium equations which, in turn, can be reduced by substitution and successive derivatives to Morley's modified form of Donnell's equations. The purpose of this Note is to report the basis and form of Koiter's modified energy functional.

Modified Energy Functional

For a circular cylindrical shell of radius R and uniform thickness h, the strain energy for small finite deflections as given by Koiter[5] is

$$P_2[\mathbf{u}] = \int \{[Eh/2(1-\nu^2)R^2][u'^2 + (v^{\cdot} + w)^2 + 2\nu u'(v^{\cdot} + w) + \tfrac{1}{2}(1-\nu)(u^{\cdot} + v')^2] + \quad \text{(line 1)}$$
$$+ [Eh/24(1-\nu^2)R^4][w''^2 + (w^{\cdot\cdot} - v^{\cdot})^2 + 2\nu w''(w^{\cdot\cdot} - v^{\cdot}) + 2(1-\nu)(w' + \tfrac{1}{4}u^{\cdot} - \tfrac{3}{4}v')^2]\}dS \quad (1)$$
(line 2)

where $\mathbf{u}$ is a generalized displacement field and the actual displacements and their derivatives are defined in the Nomenclature. In view of the length and complexity of this expression and the influence it exerts upon subsequent expressions which can be generated from it by variational techniques, there is some desirability in introducing modifications which will lead to the ultimate solution of any particular problem with a minimum of effort. Such modifications can be accomplished by the addition or subtraction of negligibly small terms, provided a proper basis for negligibility can be established.

In general, the only such basis would be the sum of all the terms in the expression. To say that any term is negligible because it is small compared to some other term overlooks the possibility that there may be yet other terms containing the same displacement derivatives as, of the same order of magnitude as, and of opposite sign from, that term used for comparison. When combined, these larger terms could produce a sum which was not much larger than the term one is trying to show is negligible.

However, since Eq. (1) is a strain energy expression for which the sum of all the terms must be positive for stable equilibrium, it is possible to subdivide the expression into groups of terms, the sums of which are, by themselves, positive. The first such subdivision would naturally be lines 1 and 2 taken separately since they represent strain energy due to membrane and bending action, respectively. It will further be noted that the first three terms in both lines 1 and 2 are of the form

$$b^2 + 2\nu bc + c^2 \quad (2)$$

which must be positive irrespective of the values of b and c (as long as ν is less than 1). The remaining term in each line must also be positive since it is a squared term. We now have four independent bases for establishing the negligibility of terms.

For example, a term such as $[Eh^3/24(1-\nu^2)R^4](u^{\cdot} + v')^2$ can be compared to the last term of line 1 and found to be in the ratio $h^2/6(1-\nu)R^2$, which for thin shells may properly be taken as negligible. On the other hand, the negligibility of a term containing v'^2, by itself, could not be determined because no usable basis for comparison is available.

As another example, a term such as $[Eh^3/24(1-\nu^2)R^4](v^{\cdot} + w)^2$ must be compared to the sum of the first three terms of line 1, leading to a ratio of the type $(h^2/12R^2)c^2/(b^2 + 2\nu bc + c^2)$ which will be maximized when the ratio $c/b = -1/\nu$. One concludes then that the maximum size of our hypothetical term $Eh^3(v^{\cdot} + w)^2/24(1-\nu^2)R^4$ must be less than $h^2/12R^2(1-\nu^2)$ multiplied by all the terms in Eq. (1) and is therefore negligible.

For establishing the negligibility of a term involving a product such as $[Eh^3/24(1-\nu^2)R^4]u'(w^{\cdot\cdot} - v^{\cdot})$ where terms from both lines 1 and 2 must be included in the comparison, a pyramiding technique is used. In this case a ratio of the type

$$(h^2/12R^2)be/[b^2 + 2\nu bc + c^2 + (h^2/R^2)(d^2 + 2\nu de + e^2)]$$

must be maximized but only after the two groups of terms in the denominator have been minimized relative to b or e independently.

Following similar reasoning, Koiter has suggested the addition of one more line of terms of the type in line 2 which, when compared to the sum of the lines 1 and 2, must in sum

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be less by factor of $0.7h/R$. The new line

$$[Eh^3/24(1 - \nu^2)R^4][-2(1 - \nu)u'(w^{\cdot\cdot} - v^{\cdot}) + 2(v^{\cdot} + w)(w'' + w^{\cdot\cdot} - v^{\cdot}) - \tfrac{3}{8}(1 - \nu)(u^{\cdot} + v')^2 + (1 - \nu)(u^{\cdot} + v')(w'^{\cdot} + \tfrac{1}{4}u^{\cdot} + \tfrac{3}{4}v') + (v^{\cdot} + w)^2]$$

when combined with the second original line yields the complete modified functional

$$P_2[\mathbf{u}] = \int \{[Eh/2(1 - \nu^2)R^2][u'^2 + (v^{\cdot} + w)^2 + 2\nu u'(v^{\cdot} + w) + \tfrac{1}{2}(1 - \nu)(u^{\cdot} + v')^2] + [Eh^3/24(1 - \nu^2)R^4] \times [w''^2 + w^{\cdot\cdot 2} + 2\nu w''w^{\cdot\cdot} + 2(1 - \nu)w'^{\cdot 2} + 2w(w'' + w^{\cdot\cdot}) + w^2 + 2(1 - \nu)(u'v^{\cdot} - u^{\cdot}v' + u^{\cdot}w'^{\cdot} - u'w^{\cdot\cdot} + v^{\cdot}w'' - v'w'^{\cdot})]\}dS \quad (3)$$

Conclusion

Although the aforementioned modifications produce no apparent simplification, Eq. (3) will, when load terms are included and variational methods applied, yield equilibrium equations which are relatively convenient to work with. Such equations, expressed in terms of the shell deflections and their derivatives can be combined, as is frequently done, into two fourth-order equations relating the tangential and longitudinal displacements independently to the radial displacements, and one eighth-order equation in terms of radial displacements only. The three equations thus obtained are in the form presented by Morley.

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Planar Motion of a Large Flexible Satellite

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Introduction

UTILIZING the method developed in Refs. 8 and 9 for the stability analysis of coupled rigid-elastic systems, the effect of flexible antennas upon the stability of motion of a gravity-gradient satellite is investigated. The satellite is assumed to consist of a compact rigid-body containing two antennas located at 180° from each other and in the plane of the orbit. The conditions for stability are found to include the well-known rigid-body stability criteria, and in addition, requirements on the elastic and coupled rigid-elastic motion.

The importance of the elastic degrees of freedom was discovered with the unexpected tumbling of Explorer I in 1958. This led to the work of Thomson and Reiter[1] who were the first to show, in a somewhat heuretic fashion, that elastic energy could adversely affect the motion of a spacecraft. Subsequent analysis were carried out,[2–7] but in most cases, assumptions were made which restricted the results, and the analysis techniques used were not easily extended to cases other than the one being analyzed. Recently, however, two somewhat similar methods for the stability analysis of a system made up of rigid and elastic elements were developed simultaneously by Meirovitch[8] and Budynas.[9] Both methods are based on the Lyapunov direct method and provide a general and rigorous approach for the stability analysis of mechanical systems made up of rigid and elastic elements. The details of the two approaches differ somewhat in that 1) different reference coordinate systems are used, and 2) Ref. 8 makes use of eigenvalues rather than eigenfunctions to perform the stability investigation. Meirovitch's use of eigenvalues rather than eigenfunctions, however, does avoid one of the difficulties encountered in Ref. 9, namely, that of determining the proper truncation of the modal series However, for complicated systems, these eigenvalues will have to be obtained by approximate methods or experimentally, and thus any great advantage may be lost. As an illustration of the method developed, Meirovitch[8] examines the case of a gravity-gradient stabilized satellite with spin, while Budynas[9] first studies the case of planar motion and then investigates the case of three-dimensional motion without spin.

Since a three-dimensional application of the technique has already been published[8] this paper summarizes the investigation of the planar case carried out in Refs. 9 and 10. The results of the planar case are of interest since they provide an uncomplicated comparison between the well known stability criteria of a rigid-body gravity-gradient stabilized satellite and the additional criteria imposed by considering the elastic degrees of freedom.

Analysis

Because the method for stability discussed by Meirovitch[8] is similar to the one used here, this section will only briefly recapitulate the stability method to be used. Reference 8 or 10 can be consulted for a more detailed discussion. The Hamiltonian for a coupled rigid-elastic system is given by

$$H = \sum_i \dot{q}_i \partial L_1/\partial \dot{q}_i + \int \Big(\sum_i \dot{q}_i \partial \pounds_1/\partial \dot{q}_i + \sum_j \dot{\eta}_j \partial \pounds_1/\partial \dot{\eta}_j \Big) \times dx - L \quad (1)$$

where q_i and $\dot{q}_i$ are the rigid-body's generalized position and velocity; η_j and $\dot{\eta}_j$ are the elastic displacement and the rate of change of elastic displacement, and

$$L = T - V = L_1(q_i,\dot{q}_i) + \int \pounds_1(q_i,\dot{q}_i,\eta_j,\dot{\eta}_j,x,\partial\eta_i/\partial x,\ldots)dx \quad (2)$$

In Eq. (2), L_1 is that part of the Lagrangian that can be expressed as a function of the rigid-body terms alone, and $\pounds_1$ is that part that can be expressed as a function of the coupled rigid-elastic terms.

If it is assumed that the elastic displacement of the system η is measured with respect to a rigid-body reference axes such that $\eta = \eta(x)$, then the position of the ith elastic elements, with respect to an inertial reference frame is expressible in the form

$$\mathbf{r} = \mathbf{r}_i[q_1,q_2,\ldots,q_n,x,\eta(x),t]$$

where $q_1,q_2,\ldots,q_n$ are the rigid-body generalized coordinates. In this case the kinetic energy of the system can be written

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